

Centre de Physique Théorique*- CNRS - Luminy, Case 907
F-13288 Marseille Cedex 9 - France

RELATIVISTIC ELASTOMECHANICS IS A GAUGE-TYPE THEORY

Jerzy KIJOWSKI ¹ and Giulio MAGLI ²

Abstract

A new approach to relativistic elasticity theory is proposed. In this approach the theory becomes a gauge-type theory, with the diffeomorphisms of the material space playing the role of gauge transformations. The dynamics of the elastic material is expressed in terms of three independent, hyperbolic, second order partial differential equations imposed on three (independent) gauge potentials. The relationship with the Carter-Quintana approach is discussed.

Key-Words : relativistic elasticity, variational principles, gauge theories.

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anonymous ftp or gopher : cpt.univ-mrs.fr

* Unité Propre de Recherche 7061

¹ Permanent address : Centre for Theoretical Physics, Polish Academy of Sciences
Al. Lotników 32/46; 02-668 Warsaw, Poland

² Permanent address : Dipartimento di Matematica del Politecnico di Milano
Piazza Leonardo da Vinci 32, 20133 Milano, Italy

1. Introduction

The interaction between the gravitational field and an elastic solid body became quite an important problem in astrophysical applications, since the discovery that the crust of neutron stars probably exists in the form of a solid, due to the process of crystallization of dense neutron matter (see e.g. McDermott, Van Horn & Hansen 1988). Recently, the present authors proposed a new approach to relativistic elasticity (Kijowski & Magli 1992). The aim of the present paper is to show the relationship between our approach and the theory proposed by Carter and Quintana (1972) and developed in Carter (1980). It turns out that the method of deriving the dynamical equations of the theory used by the latter authors is analogous to the derivation of (special-relativistic) Maxwell equations *via* the (general-relativistic) Hilbert variational principle, without introducing the notion of electromagnetic potential. On the other hand, our approach is analogous to the standard description of electrodynamics in terms of potentials.

In our formulation all the physical quantities (like e.g. the stress and the strain tensors, the matter current and so on) are defined in terms of first order derivatives of the potentials. This way, all the compatibility conditions of the theory are automatically satisfied. As a consequence, the dynamics can be formulated in terms of three (independent) second-order hyperbolic partial differential equations imposed on three (independent) unknown functions: the gauge potentials. This simplifies considerably the dynamical structure of the theory.

The equivalence between the two formulations is a straightforward consequence of the following, remarkable feature of the relativistic mechanics of continuous media: the symmetric and the canonical energy-momentum tensors of the theory do coincide (due to the convention which is generally used, they actually coincide up to a sign - see e.g. Jezierski & Kijowski 1991). It is well known that this is not the case for a general relativistic field theory, like e.g. electromagnetism, although the relationship between the two tensors is well understood (see e.g. Kijowski & Tulczyjew 1979). In the particular

case of relativistic elasticity, the simplest way to understand the equivalence between the two tensors is based on the following observation. Relativistic elasticity interacting with the gravitational field may be regarded as the theory of two symmetric tensor fields: the physical metric g and the *material metric* h (see section 2). As will be seen in the sequel, the Lagrangian of the theory depends on both metrics *via* their combination $g^{\mu\nu}h_{\nu\lambda}$ only. Hence, its variations with respect to both metrics coincide (up to a sign).

2. Kinematics

Let \mathcal{M} be the general-relativistic space-time equipped with a pseudo-riemannian metric tensor $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) of signature $(-, +, +, +)$. Suppose that \mathcal{M} (or an open domain $\mathcal{O} \subset \mathcal{M}$) is filled with a continuous material. The collection of all the trajectories of the material points may be considered as a 1-dimensional foliation of \mathcal{O} . This foliation can be described, as usual, by a normalized, time-like, future oriented vector field u^μ , tangent to the trajectories and called the *four-velocity*:

$$u^\mu u_\mu = -1. \quad (1)$$

To complete the description of the configuration of the material one has to include the information about such quantities as the matter density, internal strains, temperature etc., according to the physical character of the material. In the case of a barotropic and isotropic

elastic body, which we are going to describe in the present paper, the complete information about the configuration is carried by its internal *material metric*. It describes the “would be” rest-frame distances between adjacent molecules of the body, if the corresponding infinitesimal portion of the body had been extracted from the rest of the material and left in the perfectly relaxed state. Mathematically, the material metric will be described by a symmetric tensor field $h_{\mu\nu}$.

The “material distance” between points belonging to the same trajectory has to vanish, because they correspond to the same molecule of the body. Hence, $h_{\mu\nu}$ has to be orthogonal to the velocity:

$$h_{\mu\nu}u^\mu = 0 . \quad (2)$$

This implies that h has the signature $(0, +, +, +)$, and therefore it may assume the value of any non-negative, symmetric tensor, having precisely *one* vanishing eigenvalue, and such that the corresponding eigenvector is time-like. At each point of \mathcal{M} there is a 9-parameter family of such objects. Observe that the information about the velocity u is already contained in h . Indeed, u may be defined as the unique time-like, future-oriented, normalized eigenvector of h . We conclude that h carries the entire information about the configuration of the physical system in question. The dynamical equations of relativistic elasticity theory may be formulated as first order differential equations imposed on the quantity h .

We are going to describe only materials without memory, i.e. such that the material distance between two adjacent particles remains constant during the evolution. More precisely, this condition means the following: extracting an infinitesimal portion of the material and letting it relax leads to the same distance between the particles, independently of the moment at which the portion has been extracted. Mathematically, this means that the metric h is “frozen” in the material, i.e. its Lie derivative with respect to u vanishes:

$$\mathcal{L}_u h_{\mu\nu} = 0 , \quad (3)$$

where

$$\begin{aligned} \mathcal{L}_u h_{\mu\nu} &:= u^\lambda \nabla_\lambda h_{\mu\nu} + h_{\mu\lambda} \nabla_\nu u^\lambda + h_{\nu\lambda} \nabla_\mu u^\lambda = \\ &= u^\lambda \partial_\lambda h_{\mu\nu} + h_{\mu\lambda} \partial_\nu u^\lambda + h_{\nu\lambda} \partial_\mu u^\lambda . \end{aligned} \quad (4)$$

As u is also a function of h , equation (3) has to be considered as an identity imposed on h alone; only those h describe physically admissible configurations of the material, which fulfill this condition. Due to the identity

$$u^\mu \mathcal{L}_u h_{\mu\nu} = \mathcal{L}_u (u^\mu h_{\mu\nu}) - h_{\mu\nu} \mathcal{L}_u u^\mu \equiv 0 ,$$

there are only 6 independent conditions in (3), imposed on the 9 independent components of h . We conclude that the configurations of an elastic material are described by 3 independent functions (degrees of freedom) defined implicitly by the identities (3).

3. Relativistic strain tensor

The material is locally relaxed at a point $x \in \mathcal{M}$ if and only if the physical distances described by the metric g coincide with the material distances described by h . This happens if the material metric coincides with the physical metric on the subspace orthogonal to u , i.e. if the following equation is satisfied at x :

$$h_{\mu\nu} = E_{\mu\nu} , \quad (5)$$

where by E we denote the orthogonal projector

$$E_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu .$$

In a generic situation h is not equal to E . The bigger is the difference between them, the stronger is the state of strain of the material. There are many ways to measure this state; one possibility is to introduce their difference (Cattaneo 1973, Maugin 1978):

$$\Sigma_{\mu\nu} := \frac{1}{2}(E_{\mu\nu} - h_{\mu\nu}) . \quad (6)$$

Another description has been proposed by the present authors (see Kijowski & Magli 1992) in terms of the quantity

$$S := \frac{1}{2} \log K \quad (7)$$

where

$$K_\nu^\mu := g^{\mu\lambda}(h_{\lambda\nu} - u_\lambda u_\nu) .$$

Such strain tensors are both orthogonal to u and vanish for the locally relaxed state. There is a one-to-one correspondence between them. Therefore, both descriptions are equivalent. From the theoretical point of view, the description (7) is somewhat preferable since S is free to assume any value of a symmetric tensor orthogonal to u , whereas Σ is subject to a rather involved matrix inequality ($h = E - 2\Sigma \geq 0$). In the linearized version of the theory, both descriptions obviously coincide.

The internal elastic energy accumulated in each portion of the body is a function of its deformation, described by its state of strain. Hence, the energy depends upon both the physical metric g and the material metric h via their combination S (or Σ).

4. Carter-Quintana variational principle

A kinematically admissible h may describe a real physical situation if and only if it satisfies also the dynamical equations of the theory, which can be derived from a variational principle. For this purpose Carter and Quintana consider a bigger physical system, composed of both the elastic body and the gravitational field interacting with it. The total Lagrangian density of such a system is:

$$\Lambda = -\sqrt{-g} \left(\frac{1}{16\pi} R + \epsilon \right) ,$$

where ϵ is the rest-frame energy density of the material and R is the scalar curvature of the space-time. Keeping the material configuration h fixed and varying Λ with respect to the gravitational field g we obtain Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad (8)$$

where $T_{\mu\nu}$ is the symmetric energy-momentum tensor:

$$T_{\mu\nu} := 2 \frac{\partial \epsilon}{\partial g^{\mu\nu}} - \epsilon g_{\mu\nu} . \quad (9)$$

In this approach the dynamical equations for h arise only as the compatibility conditions

$$\nabla_\mu T^{\mu\nu} = 0 \quad (10)$$

of (8) with the Bianchi identities, satisfied by the Einstein tensor G . Among eqs. (10), only three are independent (elasticity theory has three degrees of freedom, as we have seen) and in fact in the Carter and Quintana paper it is shown that the equation

$$u_\nu \nabla_\mu T^{\mu\nu} = 0$$

holds identically. The proof is rather involved and requires the introduction of the concept of *convected derivative*.

But the fundamental drawback of the Carter-Quintana method of deriving dynamical equations is that elastodynamics and geometro-dynamics can not be separated. Indeed, the gravitational field can not be given *a priori*, as e.g. in special relativity. The theory of “test” elastic bodies in the flat Minkowski space is therefore excluded, because the space-time metric has always to be considered as a dynamical variable. Also the non-relativistic theory, which we know to be a Lagrangian theory (see e.g. Sommerferld 1950) is automatically excluded. Moreover, it is very difficult to impose upon the general theory the existence of symmetries in order to handle specific problems, as the equilibrium of axisymmetric solid stars (see e.g. Carter 1973, Carter & Quintana 1975, Quintana 1976, Priou 1992).

To illustrate the Carter-Quintana approach let us consider classical, Maxwell electrodynamics. Here, the field configurations are described by the electromagnetic, skew-symmetric tensor $F_{\mu\nu}$. In our example F plays a role analogous to that of h in elastodynamics. The first pair of Maxwell equations:

$$\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0 \quad (11)$$

can be considered as the kinematical condition, analogous to (3). We will show that the remaining dynamical Maxwell equations can be obtained from the general-relativistic procedure, analogous to the Carter-Quintana approach. For this purpose, consider the Lagrangian of the system composed of both the electromagnetic and the gravitational field:

$$\Lambda = -\sqrt{-g} \left(\frac{1}{16\pi} R + \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \right) . \quad (12)$$

Keeping the electromagnetic field F fixed and varying Λ with respect to the gravitational field g we obtain Einstein equations:

$$G_\nu^\mu = 8\pi \left(F^{\mu\rho} F_{\rho\nu} + \frac{1}{4} F^{\sigma\rho} F_{\sigma\rho} \delta_\nu^\mu \right) .$$

Now, Maxwell equations may be obtained as the compatibility conditions for the above system. Indeed, the Bianchi identities imply

$$F_{\rho\nu} \nabla_\mu F^{\mu\rho} + F^{\mu\rho} \nabla_\mu F_{\rho\nu} + \frac{1}{2} F^{\rho\sigma} \nabla_\nu F_{\rho\sigma} = 0 .$$

Using the skew-symmetry of F and the kinematical equations (11), it is easy to see that the last two terms cancel. Hence, for a generic, non singular F , we obtain the second pair of Maxwell equations

$$\nabla_\mu F^{\mu\rho} = 0 . \quad (13)$$

Of course, the above approach to electrodynamics, although equivalent to the standard one, is very inconvenient if one wants to describe the electromagnetic field in a given space-time geometry (e.g. in Minkowski space).

5. Relativistic elasticity in terms of potentials.

To obtain the standard variational principle for the Maxwell field (not necessarily coupled with the gravitational field), one introduces the electromagnetic potential A_μ , such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \quad (14)$$

In terms of the potential, electromagnetism becomes a gauge theory:

equations (11) are automatically satisfied and equations (13) become second-order dynamical equations for the potentials. Such equations can be obtained directly from the variation of the electromagnetic Lagrangian with respect to the potentials. There is no need to couple electrodynamics to gravity, although it is possible. In the latter case the total Lagrangian is again equal to (12).

In the case of elastomechanics, we propose a similar approach. As “potentials” of the theory we take the collection of all the idealized “molecules” of the material, organized in an abstract 3-dimensional differential manifold Z , called the *material space*. The space-time configuration of the material is completely described if we specify a mapping $\mathcal{G} : \mathcal{M} \rightarrow Z$. To a given point of the physical space-time (i.e. to a given point of the space and a given instant of time) the mapping assigns the “molecule” of the material which passes through that particular point at that particular time. Given a coordinate system (ξ^a) ($a = 1, 2, 3$) in Z and a coordinate system (x^μ) in \mathcal{M} , the configuration of the material is described by three functions $\xi^a = \xi^a(x^\mu)$, which play a role similar to that of the potentials $A_\nu = A_\nu(x^\mu)$ in electrodynamics.

The above potentials may be used for the description of any continuous material (see e.g. Kijowski, Pawlik & Tulczyjew 1979 and Kijowski, Smólski & Górnicka 1990 for the description of thermo-hydrodynamics). The particular case of a barotropic, elastic material requires the use of a metric structure in the material space (Hernandez 1970, Glass & Winicour 1972, Kijowski & Magli 1992). We assume, therefore, that the space Z is equipped with a riemannian (positive) metric γ_{ab} , the *material metric*. This metric is frozen in the material and *is not* a dynamical object of the theory. It is given *a priori* for a given material, like e.g. the equation of state for a fluid. To understand its physical meaning consider an infinitesimal portion of the material. This portion will tend spontaneously to a relaxed state when no external forces act on it. The metric γ may now be defined as describing the rest-frame space distances between adjacent “molecules”, measured in such a *locally* relaxed state. There are, however, materials possessing no *globally* relaxed state. This happens if Z is not isometric to any hypersurface of \mathcal{M} , e.g. when Z is curved and \mathcal{M} is flat. We see that materials with internal stresses can also be described in this way.

Given the space-time configuration, we introduce the space-time version h of the material metric as the geometric pull-back of the metric γ from Z to \mathcal{M} , i.e.

$$h := (d\xi)^* \gamma .$$

In terms of coordinates we have:

$$h_{\mu\nu} := \gamma_{ab} \xi^a_{\mu} \xi^b_{\nu} , \quad (15)$$

where we denote $\xi^a_{\mu} := \partial_{\mu} \xi^a$. The 3×4 matrix $(\partial_{\mu} \xi^a)$ is called the *relativistic deformation gradient*.

Hence, similarly to electrodynamics, the physical field has been expressed in terms of the derivatives of the potentials: formula (15) in elastomechanics is analogous to formula (14) in electrodynamics. It solves *automatically* the kinematical condition (3), just as (14) solves automatically the kinematical condition (11) in electrodynamics. Indeed, we have:

$$\begin{aligned} \mathcal{L}_u h_{\mu\nu} &= u^{\lambda} \partial_{\lambda} h_{\mu\nu} + h_{\mu\lambda} \partial_{\nu} u^{\lambda} + h_{\nu\lambda} \partial_{\mu} u^{\lambda} = u^{\lambda} (\partial_{\lambda} h_{\mu\nu} - \partial_{\nu} h_{\mu\lambda} - \partial_{\mu} h_{\nu\lambda}) \\ &= u^{\lambda} [(\partial_{\lambda} \gamma_{ab}) \xi^a_{\mu} \xi^b_{\nu} + \gamma_{ab} \xi^a_{\mu\lambda} \xi^b_{\nu} + \gamma_{ab} \xi^a_{\mu} \xi^b_{\nu\lambda} \\ &\quad - (\partial_{\nu} \gamma_{ab}) \xi^a_{\mu} \xi^b_{\lambda} - \gamma_{ab} \xi^a_{\mu\nu} \xi^b_{\lambda} - \gamma_{ab} \xi^a_{\mu} \xi^b_{\lambda\nu} \\ &\quad - (\partial_{\mu} \gamma_{ab}) \xi^a_{\nu} \xi^b_{\lambda} - \gamma_{ab} \xi^a_{\nu\mu} \xi^b_{\lambda} - \gamma_{ab} \xi^a_{\nu} \xi^b_{\lambda\mu}] . \end{aligned} \quad (16)$$

To prove that the above expression vanishes identically, let us first observe that the deformation gradient is automatically orthogonal to the velocity vector:

$$u^{\mu} \xi^a_{\mu} \equiv 0 ,$$

because u is tangent to the trajectories of the material molecules, i.e to the lines given by the equation $\xi^a = \text{const}$. This implies also:

$$u^{\lambda} (\partial_{\lambda} \gamma_{ab}) = u^{\lambda} \xi^c_{\lambda} \frac{\partial \gamma_{ab}}{\partial \xi^c} \equiv 0 ,$$

and the other terms in (16) cancel, which ends the proof.

The theory is, of course, invariant with respect to reparametrizations of the material space, which play the role of gauge transformations. Therefore, the fields ξ^a may be regarded as gauge potentials for the “elasticity field” h . They describe the three degrees of freedom of the system in an explicit way.

The group of gauge transformations of the entire theory (elasticity interacting with gravity) is therefore the product of the group of space-time diffeomorphisms (which is the gauge group of general relativity) by the group of diffeomorphisms of the material space.

6. Dynamics

The physical laws describing the mechanical properties of the elastic material can now be formulated in terms of a system of second order hyperbolic partial differential equations for the 3 unknown fields ξ^a . The equations can be obtained from the Lagrangian

$$\Lambda = -\sqrt{-g} \epsilon . \quad (17)$$

considered as a function of the fields (ξ^a) and their first derivatives. They assume the form of the Euler–Lagrange equations:

$$\partial_{\mu} \frac{\partial \Lambda}{\partial \xi^a_{\mu}} - \frac{\partial \Lambda}{\partial \xi^a} = 0 . \quad (18)$$

We are going to prove the equivalence of the above equations with the equations (10) of Carter-Quintana. For this purpose, we consider the canonical energy-momentum tensor of our theory:

$$\mathcal{T}_\nu^\mu := \frac{1}{\sqrt{-g}} \left(\frac{\partial \Lambda}{\partial \xi_\mu^a} \xi_\nu^a - \delta_\nu^\mu \Lambda \right) . \quad (19)$$

Due to the standard Nöther argument, the Euler-Lagrange field equations imply the energy-momentum conservation:

$$\nabla_\mu \mathcal{T}^{\mu\nu} = 0 . \quad (20)$$

The same argument shows (see Kijowski & Magli, 1992) that the identity

$$u_\nu \nabla_\mu \mathcal{T}^{\mu\nu} \equiv 0$$

holds automatically because of the gauge invariance of the theory. Hence, in the case of elastomechanics there are only 3 independent equations among the 4 equations (20). To prove their equivalence with the 3 Euler-Lagrange equations (18), it is now sufficient to show that both energy-momentum tensors coincide up to a sign:

$$\mathcal{T}_{\mu\nu} \equiv -T_{\mu\nu} . \quad (21)$$

In fact, this is a particular case of the general Belinfante-Rosenfeld theorem (Belinfante 1940, Rosenfeld 1940). The theorem uses the invariance of the Lagrangian with respect to the space-time diffeomorphisms. This means that it may be easily proved in framework which is more general than the one used so far in the present paper, namely, we are free to consider anisotropic elastic bodies as well. In order to introduce the variational principle for such bodies, we define the following tensor in the material space:

$$\Theta^{ab} := g_{\mu\nu} \xi_\mu^a \xi_\nu^b ,$$

from the geometrical point of view, such a tensor is the “image” of the physical metric in the material space, i.e. Θ^{ab} plays in Z the same role played by $h_{\mu\nu}$ in \mathcal{M} . For a general, possibly anisotropic body, we assume the energy density to be a function of the whole Θ^{ab} :

$$\epsilon = \epsilon(\Theta^{ab}, \xi) .$$

The particular case of isotropic bodies corresponds to an internal energy which is a function of the invariants of Θ^{ab} only; of course, the invariants of Θ^{ab} coincide with the corresponding invariants of $h_{\mu\nu}$, like e.g.:

$$\Theta_a^a = \gamma_{ab} \Theta^{ab} = \gamma_{ab} \xi_\mu^a \xi_\nu^b g^{\mu\nu} = h_{\mu\nu} g^{\mu\nu} = h_\mu^\mu .$$

According to the definition (19), we have:

$$-\mathcal{T}_\nu^\mu = \frac{\partial \epsilon}{\partial \xi_\mu^a} \xi_\nu^a - \delta_\nu^\mu \epsilon = \frac{\partial \epsilon}{\partial \Theta^{cd}} \frac{\partial \Theta^{cd}}{\partial \xi_\mu^a} \xi_\nu^a - \epsilon \delta_\nu^\mu = 2 \frac{\partial \epsilon}{\partial \Theta^{ac}} g^{\mu\alpha} \xi_\alpha^a \xi_\nu^b - \epsilon \delta_\nu^\mu ,$$

On the other hand, due to definition (9), we have

$$T_{\mu\nu} = 2 \frac{\partial \epsilon}{\partial g^{\mu\nu}} - \epsilon g_{\mu\nu} = 2 \frac{\partial \epsilon}{\partial \Theta^{ab}} \frac{\partial \Theta^{ab}}{\partial g^{\mu\nu}} - \epsilon g_{\mu\nu} = 2 \frac{\partial \epsilon}{\partial \Theta^{ab}} \xi_\mu^a \xi_\nu^b - \epsilon g_{\mu\nu} , \quad (22)$$

which ends the proof of the Belinfante–Rosenfeld identity (21). It is worthwhile to note that the first term in (22) is automatically orthogonal to the velocity. Thus, the energy–momentum tensor assumes the canonical form

$$T_{\mu\nu} = \epsilon u_\mu u_\nu + p_{\mu\nu}$$

where the *pressure tensor* $p_{\mu\nu}$ is orthogonal to the velocity and is given by

$$p_{\mu\nu} := 2 \frac{\partial \epsilon}{\partial \Theta^{ab}} \xi^a_{\mu} \xi^b_{\nu} - \epsilon E_{\mu\nu} \ .$$

7. Concluding remarks

In principle, the theory presented here follows the lines of the approach to relativistic elasticity proposed by Cattaneo (1973). There is, however, a fundamental difference: Cattaneo describes the

configuration of the material always with respect to a fixed, static reference configuration. Choosing a space–like surface $\{t = \text{const}\}$ of this reference configuration we obtain a specific representation of our material space Z . We find it, however, more natural to use only the abstract, metric structure of Z , instead of working always with two different space-times: one to describe the actual configuration and another to describe the reference configuration. Both theories are equivalent in the particular case of flat materials (no internal–frozen stresses, flat material metric) and fixed gravitational field (e.g. special relativity). In order to study the interaction between gravity and elasticity in Cattaneo’s formulation, it is necessary to follow an *ad hoc* approach, which consists in defining a reference state of the material when the gravity is hypothetically “switched off” (Newton constant set equal to zero), and then in studying the evolution during a process of adiabatic restore of the coupling constant to its own value (Cattaneo 1973, Cattaneo & Gerardi 1975). Mathematically, this is a nice way to overcome the difficulties of defining a reference state. From the physical point of view, however, switching off the gravity is poorly justified.

An approach similar to ours has also been given by Maugin (1978). However, this author works in a “direct” picture rather than in the “inverse” picture, proposed in our paper. The difference between these two pictures consists in inverting the role of the space parameters (x^k) and the fields (ξ^a) . In the “direct” picture the configuration of the material is described by the 3 fields x^k depending upon 4 independent parameters (x^0, ξ^a) . For this purpose a (3+1)-decomposition of the space-time has to be chosen, which breaks the explicit relativistic invariance of the theory. However, the theory proposed by Maugin remains relativistic-covariant and its results are independent of a specific choice of the (3+1)-decomposition.

The advantage of our approach consists in the fact that we really eliminate all the constraints in a fully relativistic-invariant way, reducing the number of independent degrees of freedom to the three independent functions ξ^a . We hope this approach to be useful for specific applications like e.g. the description of the equilibrium and the oscillations of neutron stars crusts in a general relativistic context. Some steps in this direction have already been done (Magli & Kijowski 1992, Magli 1993a,b). Moreover, as will be shown in a forthcoming paper (Kijowski & Magli 1993), our formulation naturally leads to the hamiltonian version of the theory, the canonical variables being the fields ξ^a and their conjugate momenta $\pi_a = \partial\Lambda/\partial\dot{\xi}^a$. The Poisson bracket between them assumes its canonical, delta–like form. The

above hamiltonian structure is common for any (relativistic or non-relativistic) mechanics of continuous media (see also Kijowski, Smólski & Górnicka 1990 and Jezierski & Kijowski 1991). It is given by a standard Legendre transformation, once the Lagrangian (17) is expressed in terms of the (independent) variables (ξ^a) and their first (unconstrained) derivatives (see also Kijowski & Tulczyjew 1979).

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